Weierstrass approx Thm
beginning part: Functional

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Prolegomenon

Fix X, a compact Hausdorff space, and let $\mathbf{C}(X \to \mathbb{R}) =: \mathbf{C}(X)$ be the space of continuous fncs $X \to \mathbb{R}$, topologized by the **supremum norm**. Our goal is to prove WAT (Weierstrass Approximation Thm), below. But first, we need:

1: *Defn.* A non-void subset $\mathcal{L} \subset \mathbf{C}(X)$ is a ***good lattice*** if: $\mathcal{L}$ is a lattice under the pointwise Min and Max operations, i.e, the $\leqslant$ relation. [So $\mathcal{L}$ is a *sublattice* of $(\mathbf{C}(X), \mathrm{Min}, \mathrm{Max})$.] Further

a: $\mathcal{L}$ ***separates points***. (I.e, for all distinct $y, z \in X$ there exists $g \in \mathcal{L}$ with $g(y) \neq g(z)$.)

b: For all $\sigma, \tau \in \mathbb{R}$ and $f \in \mathcal{L}$:

$\sigma \cdot f() \in \mathcal{L}$. [Scaling.]

And $\tau + f() \in \mathcal{L}$. [Vertical translation.]

With a particular good lattice in mind, I'll call its members the ***good functions***. For free we get:

c: Our $\mathcal{L}$ owns all the constant functions.

For by hyp. $\mathcal{L}$ is non-empty. Take an $f \in \mathcal{L}$ and scale it by $\sigma{=}0$, to conclude that $\mathcal{L}$ owns the zero-fnc. Now vertical translation gives us all the constant-fncs. □

2: Two-point Proposition. *In $\mathcal{L}$, a good lattice: For each function $T : X \to \mathbb{R}$, and each pair of points $y, z \in X$, there is a good function g st.*

$$g(y) = T(y) \quad \text{and} \quad g(z) = T(z)\,. \qquad \diamondsuit$$

Proof. If $y = z$, then $T(y) = T(z) =: C$ is a common value. Let g be the constant-fnc, $[x \mapsto C]$.

Otherwise, $y \neq z$. Since $\mathcal{L}$ separates points, there is a good f with $f(y) \neq f(z)$. By adding a constant, (1b), WLOG $f(z) = 0$. By scaling, WLOG $f(y) = 1$. Using numbers $Y := T(y)$ and $Z := T(z)$, the function

$$g() \quad := \quad Z \;+\; [Y - Z] \cdot f()$$

is good. Moreover, $g(y) = Z + [Y - Z] \cdot 1 \overset{\text{note}}{=\!=} Y$ and $g(z) = Z + 0$ equals Z, as desired. ♦

3: Lattice Lemma. *Each good lattice $\mathcal{L}$ is sup-norm dense in $\mathbf{C}(X \to \mathbb{R})$.* ◊

Proof. Fix a target fnc $T \in \mathbf{C}(X)$. Fixing a posreal ε, we will produce a good h with

$$\text{3a:} \qquad T() \;<\; h() \;<\; 2\varepsilon + T()\,.$$

It suffices, for each point $z \in X$, to be able to produce a good function f_z having

$$\text{3b:} \qquad \begin{aligned} f_z() \;&<\; \varepsilon + T() \quad \text{and} \\ f_z(z) \;&=\; T(z)\,. \end{aligned}$$

Why? Well, replacing $f_z()$ by $\varepsilon + f_z()$ keeps f_z good, using (1b). And now

$$\dagger\text{:} \qquad \begin{aligned} f_z() \;&\leqslant\; 2\varepsilon + T() \quad \text{with} \\ f_z(z) \;&>\; T(z)\,. \end{aligned}$$

Consequently, the following "f Over T" set,

$$\mathcal{O}_z \;:=\; \left\{ x \in X \;\middle|\; f_z(x) > T(x) \right\},$$

is open, and owns z. Thus $\{\mathcal{O}_z \mid z \in X\}$ is an *open cover* of X. But X is compact, so there is a finite list where $\mathcal{O}_{z_1} \cup \mathcal{O}_{z_2} \cup \ldots \cup \mathcal{O}_{z_J}$ is all of X. The Upshot: The *good* fnc

$$h \;:=\; \mathrm{Max}(f_{z_1}, f_{z_2}, \ldots, f_{z_J})$$

automatically fulfills (3a).

Creating (3b). Fix z. For each $y \in X$, proposition (2) gives us a fnc g_y with

$$\begin{aligned} g_y(y) &= T(y) \quad \text{and} \\ g_y(z) &= T(z)\,, \end{aligned}$$

and which is good. The "g Under T" set

$$\mathfrak{U}_y \;:=\; \Big\{x \in X \;\Big|\; g_y(x) < \varepsilon + T(x)\Big\}$$

is open, and owns y. So $\{\mathfrak{U}_y\}_{y \in X}$ is an open cover of X. Some finitely many $\mathfrak{U}_{y_1} \cup \mathfrak{U}_{y_2} \cup \ldots \cup \mathfrak{U}_{y_K}$ cover X. Therefore the *good* fnc

$$f_z \;:=\; \text{Min}(g_{y1}, g_{y2}, \ldots, g_{yK})$$

automatically fulfills (3b). ♦

We are now ready for the Weierstrass Approximation Theorem.

4: WAT Thm. *On a compact Hausdorff space X, consider a (real) sub-vectorspace $\mathbf{V} \subset \mathbf{C}(X)$. Suppose $\mathbf{V}$ satisfies the following:*

a: *Subspace $\mathbf{V}$ separates points of X.*

b: $f \in \mathbf{V} \implies f^2 \in \mathbf{V}$. *($f$ times f.)*

c: $\mathbf{1}_X \in \mathbf{V}$. *("The constants are in $\mathbf{V}$.")*

Then $\mathbf{V}$ is an algebra under pointwise multiplication $[f, g \in \mathbf{V} \implies f \cdot g \in \mathbf{V}]$ and is sup-norm dense in $\mathbf{C}(X)$. ♢

Proof of WAT. With $f, g \in \mathbf{V}$ then $f+g \in \mathbf{V}$. Since $\mathbf{V}$ is sealed under squaring,

$$\mathbf{V} \ni \tfrac{1}{2} \cdot \Big[[f+g]^2 - f^2 - g^2\Big] \overset{\text{note}}{=\!=} f \cdot g\,.$$

Thus $\mathbf{V}$ *is* an algebra of fncs.

On $\mathbf{C}(X)$, addition is cts [Exer. 1a], so the $\mathbf{C}(X)$-closure of $\mathbf{V}$ is a sub-vectorspace [Exer. 1b]. This closure automatically satisfies (4a) and (4c).

Since X is compact, each $f \in \mathbf{C}(X)$ is bounded. It follows that the squaring map $[f \mapsto f^2]$ is sup-norm cts [Exer. 2a]. It further follows [Exer. 2b] that the squaring property (4(b)) holds for the closure of $\mathbf{V}$. So WLOG $\boxed{\mathbf{V} \text{ is closed}}$.

Let's show that $\mathbf{V}$ is a good lattice. Properties (1a,b,1) are automatic. Calling the functions in $\mathbf{V}$ ***nice***, our goal is

G1: f, g each nice $\implies$ Max$\{f, g\}$ is nice.

(Since Min$\{f, g\}$ equals the negative of Max$\{-f, -g\}$, we get Min$\{\}$ for free.) Evidently

$$\text{Max}\{f, g\} \;=\; \tfrac{1}{2}\Big[f+g \;+\; |f-g|\Big]\,.$$

This will be in $\mathbf{V}$ if $|f-g|$ is. So our goal is to show that

G2: *Algebra $\mathbf{V}$ is sealed under the absolute-value operation.*

Note that $|f| = \sqrt{f^2}$, for an arbitrary fnc $f: X \to \mathbb{R}$ in $\mathbf{V}$. Since f^2 is nice, need but show the following, where we've renamed f^2 to f:

G3: *If $f \geqslant 0$ is nice, then $\sqrt{f}$ is nice.*

Since X is compact, the range of f lies inside some compact interval $J \subset \mathbb{R}$. Now show [Exer. 3] that the-below Square-root Theorem is sufficient to finish the proof. ♦

5: Unif-conv Composition Lemma. *Consider sets Z, Y and MSes X and Ω. For maps $f_n, g: Y \to X$, suppose $f_n \xrightarrow{\text{unif.}} g$, as $n \to \infty$. Then the following hold.*

i: *For $\beta: Z \to Y$ an arbitrary function, sequence $[f_n \circ \beta] \xrightarrow[n\to\infty]{\text{unif.}} [g \circ \beta]$.*

ii: *Suppose map $\alpha: X \to \Omega$ is uniformly continuous. Then functions $[\alpha \circ f_n]$ converge uniformly to $[\alpha \circ g]$, as $n \to \infty$.* ♢

Proof. Exercise. ♦

6: Square-root Theorem. *Fix a compact interval $J \subset [0, \infty)$. Then the sqrt-function on J is a uniform limit of polynomials on J.* ♢

Proof. By a change-of-variable, WLOG $J := [0, 1]$.
ISTShow [Exer. 4] that on J, this fnc

$$h(T) \;:=\; 1 - \sqrt{1-T}$$

is a uniform-limit of polynomials.

Thus, by our Nested uniform-convergence thm, ISTFind polynomials $P_0 \leqslant P_1 \leqslant \ldots H$ which converge pointwise to H.

(Rest of proof in class. Use Verhulst diagrams.) ♦

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